Math 3235 Probability Theory

09/20/22

Family of discrete r.v.

e.g. X, Y, Z

or $X_i \quad i = 1 \dots N$

$$p(x, y, z) = \mathbb{P}(X = x \ \& \ Y = y \ \& \ Z = z)$$

joint p.m.f of X, Y, Z

similarly

$$p(x_1, \dots, x_N) = \mathbb{P}(X_1 = x_1 \ \& \ \dots \ \& \ X_N = x_N)$$

joint p.m.f of the X_i

Marginals:

$$p_{X,Y}(x, y) = \sum_z p(x, y, z)$$

x, y - marginal

$$p_X(x) = \sum_{y, z} p(x, y, z)$$

x - marginal

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$W = f(X, Y, Z)$ is a r.v.

$$P_W(w) = \sum_{(x,y,z) \in f^{-1}(w)} p(x, y, z)$$

$$\mathbb{E}(W) = \sum_{x,y,z} f(x,y,z)\, p(x,y,z)$$

$$\mathbb{E}\left(\sum_{i=1}^{N} a_i X_i\right) = \sum_{i=1}^{N} a_i \mathbb{E}(X_i)$$

$$a_i \in \mathbb{R}$$

If X_i are positive and $a_i \geq 0$

Then I can take $N = \infty$ above.

Independence

$$p(x,y,z) = P_X(x) P_Y(y) P_Z(z)$$

$\forall\, x, y, z.$

If I have a family X_i

$$p(x_1, \dots x_N) = \prod_{i=1}^{N} P_i(x_i)$$

if you have a family of X_i such that

a) all X_i have the same distribution

b) They are independent

X_i are called i.i.d.

i.i.d. : independent and identically distributed

Ex.: coin flip.

X_i are Bernoulli

Rem: in Statistics an i.i.d. family is called a Random Sample

$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ if $X \perp\!\!\!\perp Y$

X and Y are independent

$\Updownarrow$

$\forall f, g \quad \mathbb{E}(f(X)g(Y)) =$

$\mathbb{E}(f(X))\mathbb{E}(g(Y))$

Defect from independence

$$\mathrm{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

if X, Y are independent

$\Downarrow$

$$\mathrm{cov}(X, Y) = 0$$

if $\mathrm{cov}(X, Y) = 0$ Then

X, Y are said

uncorrelated.

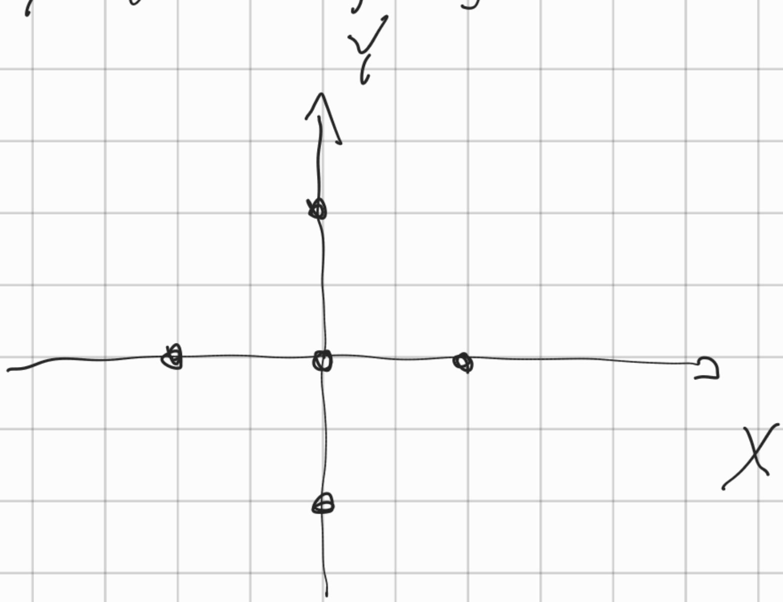
uncorrelated $\not\Rightarrow$ independent

X, Y can take the

as value

$(1, 0)$ $(-1, 0)$ $(0, 0)$

$(0, 1)$ $(0, -1)$



each with prob $\frac{1}{5}$

$P_X(1) = P_X(-1) = \frac{1}{5}$ $P_X(0) = \frac{3}{5}$

$P_Y(1) = P_Y(-1) = \frac{1}{5}$ $P_Y(0) = \frac{3}{5}$

$\mathbb{E}(X) = \mathbb{E}(Y) = 0$

$\mathbb{E}(XY) = 0$

$\Downarrow$

$\operatorname{cov}(X, Y) = 0$ uncorrelated

$p(0,0) = \frac{1}{5} \neq p_X(0)\, p_Y(0) = \frac{9}{25}$

not independent

Correlation coeff.

$$\rho_{X,Y} = \frac{\operatorname{cov}(X, Y)}{\sigma_X \sigma_Y}$$

$\rho_{aX, bY} = \rho_{X,Y}$ $\quad a, b \geq 0$

$$-1 \leq \rho_{X,Y} \leq 1$$

Sum of r.v.

X Y are jointly distributed r.v.

$$Z = X + Y$$

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X,Y)$$

if $X \perp\!\!\!\perp Y$

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$$

If X_i are i.i.d

$$\mathbb{E}(X_i) = \mu$$

$$\text{var}(X_i) = \sigma^2$$

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

$$\mathbb{E}(\bar{X}) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(X_i) = \mu$$

$$\mathrm{var}(\bar{X}) = \mathrm{var}\left(\frac{1}{N}\sum_{i=1}^{N}X_i\right) =$$

$$\frac{1}{N^2}\sum_{i=1}^{N}\mathrm{var}(X_i) = \frac{\sigma^2}{N}$$

$$\sigma_{\bar{X}} = \sigma/\sqrt{N}$$

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what about the p.m.f of $X+Y$?

$$Z = X + Y$$

$$p_Z(z) = \sum_{x+y=z} p(x,y) =$$

$$= \sum_x p(x, z-x)$$

if X and Y are independent

$$p_Z(x)\ \sum_x p_X(x)\, p_Y(z-x)$$

P_Z is the convolution of P_X and P_Y

X and Y are Poisson

λ μ

$X \perp\!\!\!\perp Y$

$Z = X + Y$, p.m.f P_Z of Z ?

$$\mathbb{P}(Z = z) = \sum_{x+y=z} \mathbb{P}(X = x \,\&\, Y = y) =$$

$$= \sum_{x+y=z} \mathbb{P}(X = x)\, \mathbb{P}(Y = y) =$$

$$= \sum_{x+y=z} \frac{\lambda^x}{x!} \frac{\mu^y}{y!} e^{-\lambda} e^{-\mu} =$$

$$= \frac{1}{z!} e^{-(\lambda+\mu)} \sum_{x+y=z} \binom{z}{x} \lambda^x \mu^y$$

$$= \frac{1}{z!} e^{-(\lambda+\mu)} (\lambda+\mu)^z$$

Z is Poissonian with par. $\lambda+\mu$

N is Poissonian

X_i are i.i.d. Bernoulli r.v. with par p

$$Y = \sum_{i=1}^{N} X_i$$

$\omega \in \Omega$ $\qquad N(\omega) \qquad X_i(\omega)$

$$Y(\omega) = \sum_{i=1}^{N(\omega)} X_i(\omega)$$

Car needing service arrival problem.

At a gas station the number of cars arriving in 1h is $N \sim Poi(\lambda)$. Each has prob p of needing service.

If Y is the number of cars that arrive and need service

$$Y = \sum_{i=1}^{N} X_i \qquad \text{where}$$

$X_i = 1$ if the i-th car needs service.

We saw that

$$Y \sim Poi(p\lambda)$$

if X_i i.i.d.

and X_i, N independent.

$Z = N - Y$ number of cars That do not need service.

$$Z = \sum_{i=1}^{N} (1 - X_i)$$

$(1 - X_i)$ are i.i.d

Bernoulli par $q = 1 - p$

$Z \sim Poi(q\lambda)$

Are Y and Z independent?

$\mathbb{P}(Y = y \ \& \ Z = z) =$

Since $Y + Z = N$

$= \mathbb{P}(Y = y \ \& \ N = y + z) =$

$= \mathbb{P}(Y = y \mid N = y + z) \, \mathbb{P}(N = y + z)$

$$\mathbb{P}(N = y+z) = e^{-\lambda} \frac{\lambda^{y+z}}{(y+z)!}$$

$\mathbb{P}(Y = y \mid N = y+z)$ is binomial in y with par $y+z$, p

$$\mathbb{P}(Y = y \mid N = y+z) = \binom{y+z}{y} p^y q^z$$

$$\mathbb{P}(X = x \ \& \ Y = y) =$$

$$\binom{y+z}{y} p^y q^z \frac{\lambda^{y+z}}{(y+z)!} e^{-\lambda} =$$

$$\binom{y+z}{y} \frac{1}{(y+z)!} = \frac{1}{y!\, z!}$$

$$p^y q^z \lambda^{y+z} = (p\lambda)^y (q\lambda)^z$$

$$e^{-\lambda} = e^{-p\lambda} e^{-q\lambda}$$

$$\mathbb{P}(Y = y \ \& \ Z = z) = e^{-p\lambda} \frac{(p\lambda)^y}{y!} \ e^{-q\lambda} \frac{(q\lambda)^z}{z!} =$$

$$= \mathbb{P}(Y = y)\,\mathbb{P}(Z = z)$$

$$\boxed{Y \perp\!\!\!\perp Z}$$

Check $\quad Y + Z = N \sim \mathrm{Poi}(\lambda).$